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Trapped waves over symmetric thin bodies

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Explicit relations are derived for the dependence of the longshore wavenumber on the wave frequency of symmetric trapped waves or edge waves travelling near the cutoff frequency over submerged horizontally symmetric thin bodies or near-vertical cliffs. Results for particular geometries are presented and shown to agree with certain explicit solutions for edge waves over sloping beaches or trapped waves over a submerged narrow shelf, or a semicircular mound on the sea bed. Similar results are obtained for thin bodies extending vertically throughout the depth in open channels or for thin cross-sections in an acoustic wave guide.

1. Introduction

Trapped waves are surface gravity waves which can travel unchanged above a long submerged horizontal cylinder but which vanish at large distances in the horizontal direction perpendicular to the generators of the cylinder. Their existence was first proved at about the same time by Ursell (1951), who constructed a trapped-wave solution over a submerged horizontal cylinder provided that the cylinder was sufficiently small, and Jones (1953) who used functional analysis arguments to prove their existence for a wide class of submerged cylinders of arbitrary but symmetric cross-sections. Recently Ursell (1987) has considerably simplified the proof using straightforward minimum-energy arguments.

Trapped waves are of considerable interest in providing examples of discrete wave frequencies in the presence of a continuous spectrum. For a given wavenumber k along the cylinder direction there exist a finite number of discrete frequencies ω corresponding to trapped waves. Jones (1953) provides bounds for the number of trapped modes in terms of the geometry of the submerged body, whilst McIver & Evans (1985) have computed trapped-wave frequencies for the submerged circular cylinder using the Ursell (1951) formulation and have shown that more trapped waves appear as the cylinder approaches the free surface.

Trapped-wave solutions over a submerged rectangular cylinder resting on the bottom have been considered by Evans & McIver (1984) who confirmed the bounds given by Jones (1953). Because of the symmetric nature of both the geometry and the solution, these solutions also describe waves travelling in the longshore direction over a shelf bounded by a vertical wall. In this context they are more commonly called edge waves and are of considerable interest to oceanographers. The simplest such edge wave exists over a uniform sloping beach and was reported by Stokes in 1846. He found a simple solution decaying exponentially out to sea with wave

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frequency $\omega/2\pi$ and wavenumber k in the longshore direction connected by the relation

$$\omega^2 = gk\cos\epsilon, \tag{1.1}$$

where ϵ is the angle of the beach to the downward vertical. Ursell (1952) showed that (1.1) was just one of a finite number of edge waves, the number increasing as the beach approached the horizontal. However, for $\epsilon < \frac{1}{3}\pi$ it is likely that the only bounded edge wave is that due to Stokes given by (1.1). A good description of edge waves in an oceanographic context is given by LeBlond & Mysak (1978).

Trapped or edge waves are always associated with a cutoff in the frequency spectrum. For fixed longshore wavenumber k there exists a value of the frequency ω_0 above which waves of all frequencies are possible which describe waves incident upon and scattered by the submerged body or beach. In this range unique reflection and transmission coefficients can be defined at sufficiently large distances. Below the cutoff frequency there exist discrete frequencies describing trapped waves which remain local to the submerged body or beach and which do not radiate energy to large distances.

In deep water the cutoff frequency $\omega_0/2\pi$ is given by $\omega_0^2 = gk$. In the case of a uniformly sloping beach, as the beach becomes steeper, $\epsilon \rightarrow 0$ in (1.1) and the edge-wave frequency approaches the cutoff frequency. In particular

$$l \equiv (k^2 - K^2)^{\frac{1}{2}} = K\tan\epsilon \sim K\epsilon, \quad \epsilon \rightarrow 0, \quad K = \omega^2/g. \tag{1.2}$$

When the beach is vertical the edge wave disappears and at the cutoff frequency a wave of constant amplitude out to sea propagates in the longshore direction.

In the problem of the submerged circular cylinder Ursell (1951) showed that there exists a trapped wave for a small enough cylinder and that the frequency of this trapped wave also approaches that of the cutoff frequency ω_0. Specifically he showed that

$$l \sim 3\pi(ka)^2 K\exp(-2kd) \tag{1.3}$$

as $ka \rightarrow 0$, where d is the depth of the centre of the cylinder which has radius a.

In both these cases an explicit relation is obtained for the wave frequency $\omega/2\pi$ (through $K = \omega^2/g$) of the trapped or edge wave when it is close to the cutoff wave frequency ω_0 (through $k = \omega_0^2/g$) and when the influence of the beach or cylinder is small.

In this paper we seek to generalize the results (1.2) and (1.3) by considering horizontal cylinders which are both symmetric and also thin in their horizontal dimensions. We shall show, by letting the thinness parameter $\epsilon \rightarrow 0$, thereby reducing the influence of the cylinders, whilst simultaneously seeking solutions for $K \sim k$, that an explicit relation between l and K or k may be obtained in terms of the shape of the body.

The analysis is carried out in the next section for finite depth of water h, where the cutoff frequency ω_{0h} is given by

$$\omega_{0h}^2 = gk\tanh kh \tag{1.4}$$

and solutions will be found for $\epsilon \rightarrow 0$ and $\kappa \sim k$, where κ is the real positive root of

$$\omega^2 = g\kappa\tanh\kappa h. \tag{1.5}$$

A further indication that an explicit relation might exist in these simultaneous limits comes from the work of Ursell (1968) who has shown that bounded head seas cannot be maintained along an infinitely long cylinder in deep water. For waves above the cutoff frequency the problem of the determination of the reflection and transmission of an obliquely incident wave by the cylinder is well defined. However, as the direction of the wave approaches grazing incidence the solution becomes unbounded in some parts of the wave field. Here we seek the limit from *below* the cutoff frequency whilst simultaneously diminishing the influence of the body.

In §3 the results are applied to a number of problems including edge waves over a nearly vertical beach, and to trapped waves over a thin ellipse in both finite depth and infinitely deep water. The result (1.2) is recovered as is the result for edge waves trapped over a narrow shelf derived on the basis of the shallow-water approximation.

Recent work by Evans & Linton (1991), McIver (1990) and Callan, Linton & Evans (1990) have shown that trapped modes can also occur when the governing equation is the two-dimensional Helmholtz equation corresponding to either an acoustic problem or a three-dimensional water-wave problem in which the depth variation can be separated out. Specifically, trapped modes can be shown to exist in the vicinity of a vertical cylinder extending throughout the water depth and placed on the centreplane of an open channel. The cylinder is symmetric with respect to both the centreplane and a vertical plane perpendicular to it and the trapped-mode solutions are antisymmetric about the centreplane.

In §4 we extend the ideas of §2 to the case when the vertical cylinder is thin and obtain results for the trapped-mode frequency in terms of the geometry of the cylinder which are in agreement with the special case of the thin rectangular cylinder treated by Evans & Linton (1991) and the thin ellipse considered by McIver (1990).

2. Formulation and solution

We choose Cartesian coordinates with x, z horizontal, y vertically downwards, $y = 0$ the undisturbed free surface and $y = h$ the rigid bottom. The submerged cylinder is symmetric about the plane $x = 0$ and is described by $x = \epsilon f(y)$ for $x \geqslant 0$. We seek a solution describing waves of frequency $\omega/2\pi$ and wavelength $2\pi/k$ travelling along the generators of the cylinder but tending to zero as $|x| \to \infty$. On the basis of linear water-wave theory we may introduce a harmonic velocity potential $\Phi(x, y, z, t)$ which we write

$$\Phi(x,y,z,t) = \mathrm{Re}\,\phi(x,y)\exp\{\mathrm{i}(kz \pm \omega t)\}. \tag{2.1}$$

Then $\phi(x, y)$ satisfies

$$(\nabla^2 - k^2)\,\phi(x,y) = 0 \quad \text{in the fluid} \tag{2.2}$$

$$K\phi + \phi_y = 0, \quad y = 0, \quad K = \omega^2/g, \tag{2.3}$$

$$\phi_y = 0, \quad y = h, \tag{2.4}$$

$$\phi \to 0, \quad x \to \infty, \quad 0 \leqslant y \leqslant h, \tag{2.5}$$

$$\phi_n = 0 \quad \text{on} \quad F(x,y) \equiv x - \epsilon f(y) = 0, \tag{2.6}$$

$$\phi(x,y) = \phi(-x,y). \tag{2.7}$$

Because of the symmetry of the problem we can restrict attention to $x \geqslant 0$ and then use (2.7) to define the potential for all x.

We can expand condition (2.6) about $x = 0$ to obtain

$$\phi_x = \epsilon(f'(y)\,\phi_y - f(y)\,\phi_{xx}) + O(\epsilon^2), \quad x = 0. \tag{2.8}$$

We now seek ϕ satisfying (2.1)–(2.8) in the strip $0 \leqslant y \leqslant h$, $x \geqslant 0$. But from Havelock's (1929) wavemaker theory we can write

$$\phi(x,y) = \mathrm{e}^{-l_0 x}\psi_0(y) + \chi(x,y), \tag{2.9}$$

where
$$\chi(x,y) = \sum_{n=1}^{\infty} A_n \,\mathrm{e}^{-l_n x} \psi_n(y). \tag{2.10}$$
Here
$$\psi_n(y) = N_n^{-\frac{1}{2}} \cos k_n(h-y), \quad n = 0,1,2,\ldots, \tag{2.11}$$
where
$$N_n = \tfrac{1}{2}(h + \sin 2k_n h/2k_n), \tag{2.12}$$
and
$$l_n = (k_n^2 + k^2)^{\frac{1}{2}}, \tag{2.13}$$
$$l_0 = (k^2 - \kappa^2)^{\frac{1}{2}}, \tag{2.14}$$
and k_n are the positive real roots of
$$K + k_n \tan k_n h = 0, \quad n = 1,2,3,\ldots, \tag{2.15}$$
with $k_0 = \mathrm{i}\kappa$, where κ is the real positive root of
$$K = \kappa \tanh \kappa h. \tag{2.16}$$
It is assumed that $k > \kappa$ so that $\phi \to 0$ as $x \to \infty$.

It can be shown that the $\{\psi_n(y)\}$, $(n = 0,1,\ldots)$ form a complete orthonormal set in $[0,h]$.

The coefficient of the first term on the right-hand side of (2.9) has been chosen as unity without loss of generality. Notice that from (2.10) the orthonormality of $\{\psi_n\}$ implies that
$$\int_0^h \chi(x,y)\,\psi_0(y)\,\mathrm{d}y = 0 \quad \text{for all } x. \tag{2.17}$$
We assume a perturbation expansion for $\chi(x,y)$ in the form
$$\chi(x,y) = \chi_0(x,y) + \epsilon\chi_1(x,y) + O(\epsilon^2), \quad \epsilon \to 0 \tag{2.18}$$
and substitute both (2.9) and (2.18) into (2.8) to obtain
$$\frac{\partial\chi_0}{\partial x}(0,y) + \epsilon\frac{\partial\chi_1}{\partial x}(0,y) = l_0\psi_0(y) + \epsilon\left\{f'(y)\left(\frac{\partial\chi_0}{\partial x}(0,y) + \psi_0'(y)\right)\right.$$
$$\left. - f(y)\left(\frac{\partial^2\chi_0}{\partial x^2}(0,y) + l_0^2\psi_0(y)\right)\right\} + O(\epsilon^2). \tag{2.19}$$
Now if in (2.19) $l_0 = O(1)$ then
$$\frac{\partial\chi_0}{\partial x}(0,y) = l_0\psi_0(y), \tag{2.20}$$
which contradicts (2.17). As explained in §1 both ϵ and l_0 need to be small if we are to obtain trapped waves.

Let
$$l_0 = \epsilon^p m_0 \quad \text{for integer } p > 0, \quad m_0 = O(1). \tag{2.21}$$
Then it follows from (2.20) that $\chi_0(x,y) \equiv 0$ and (2.19) becomes
$$\frac{\partial\chi_1}{\partial x}(0,y) = \epsilon^{p-1}m_0\psi_0(y) + \{f'(y)\,\psi_0'(y) - m_0^2 f(y)\,\mathrm{e}^{2p}\psi_0(y)\}. \tag{2.22}$$
If $p > 1$ in (2.21), then (2.22) becomes at leading order
$$\frac{\partial\chi_1}{\partial x}(0,y) = f'(y)\,\psi_0'(y), \tag{2.23}$$

which contradicts (2.17) except possibly for special forms of $f(y)$. Thus $p = 1$ and

$$\frac{\partial \chi_1}{\partial x}(0,y) = m_0 \psi_0(y) + f'(y)\,\psi_0'(y), \tag{2.24}$$

which is consistent with (2.17) provided that

$$l_0 + \epsilon \int_L f'(y)\,\psi_0(y)\,\psi_0'(y)\,\mathrm{d}y = 0, \tag{2.25}$$

where $L = \{x = 0, y : f(y) \neq 0\}$.

Trapped-wave solutions are possible if (2.25) is satisfied and the corresponding trapped mode is

$$\phi(x,y) = \mathrm{e}^{-l_0 x}\psi_0(y) + O(\epsilon), \tag{2.26}$$

which is valid throughout the fluid domain.

Now

$$\psi_0(y)\,\psi_0'(y) = -\frac{2\kappa^2 \sinh 2\kappa(h-y)}{2\kappa h + \sinh 2\kappa h} \tag{2.27}$$

and (2.25) becomes

$$l_0 = \frac{2\epsilon\kappa^2}{2\kappa h + \sinh 2\kappa h}\int_L f'(y) \sinh 2\kappa(h-y)\,\mathrm{d}y. \tag{2.28}$$

Equation (2.28) is the main result of this section. It provides a connection between the longshore wave frequency $\omega/2\pi$ of a trapped wave, through (2.14) and (2.16), when it is close to the cutoff frequency ω_{0h}, through (1.4), in terms of the shape of the thin body.

In deep water, $h \to \infty$ and the corresponding result is

$$l = 2K^2\epsilon \int_L f'(y)\,\mathrm{e}^{-2Ky}\,\mathrm{d}y, \tag{2.29}$$

where $l = (k^2 - K^2)^{\frac{1}{2}}$.

An immediate result follows from the requirement that $l, l_0 > 0$ in order for the solution to vanish at $x = \infty$. It is seen from (2.28), (2.29) that if $f'(y) \leqslant 0$ for all y then there can be no solution since the right-hand sides of (2.28) and (2.29) are negative or zero. This appears to rule out the possibility of trapped waves in the presence of a surface-piercing convex body or along an overhanging cliff of negative slope.

3. Applications

There are two types of problem to which (2.28) and (2.29) can be applied. In the first $f(y)$ is defined for all y and describes the small deviation of the boundary from $x = 0$. For example $f(y) = y$ gives a sloping beach making a small angle ϵ with the vertical. In the second $f(y)$ is chosen to be non-zero over a finite interval so that (2.28) and (2.29) determine the condition for symmetric trapped modes over the body described by $x = \epsilon f(y)$ and its reflection in $x = 0$.

3.1. *Edge waves over near-vertical beaches*

We can compare (2.29) with the result of Stokes (1846) for edge waves over a sloping beach of angle ϵ to the downward vertical.

The Stokes edge wave is

$$\varphi(x,y) = \mathrm{e}^{-kx\sin\epsilon}\,\mathrm{e}^{-ky\cos\epsilon} \tag{3.1}$$

and requires
or

$$\left.\begin{aligned} K &= k\cos\epsilon \\ l &= (k^2-K^2)^{\frac{1}{2}} = k\sin\epsilon = K\tan\epsilon. \end{aligned}\right\} \tag{3.2}$$

Now from (2.29) with $f(y) = y$ we obtain $l = K\epsilon$, in agreement with (3.2) for small ϵ. In fact the *exact* relation is recovered in this case without the approximation $\epsilon \ll 1$ as is readily seen by replacing $\epsilon f(y)$ by $y\tan\epsilon$ in (2.29), to give $l = K\tan\epsilon$ in agreement with (3.2). The deep-water mode shape corresponding to (2.29) is given from (2.26) by letting $h \rightarrow \infty$, and is

$$\varphi(x,y) = B\,\mathrm{e}^{-lx}\,\mathrm{e}^{-Ky}$$

for any constant B, in agreement with the Stokes solution.

The result for a near-vertical beach in *finite* depth follows from (2.28) with $f(y) = y$, $0 \leqslant y \leqslant h$.

Thus

$$l_0 = \frac{2\epsilon\kappa\sinh^2\kappa h}{2\kappa h+\sinh 2\kappa h}. \tag{3.3}$$

The result for an undulating beach of the form $f(y) = \sin 2\lambda y$, $\lambda > 0$, is, from (2.29),

$$l_0 = \frac{2\epsilon\lambda\kappa^3(\cosh 2\kappa h-\cos 2\lambda h)}{(\kappa^2+\lambda^2)(2\kappa h+\sinh 2\kappa h)} \rightarrow \frac{2\epsilon\lambda K^3}{K^2+\lambda^2} \quad \text{as} \quad h\rightarrow\infty. \tag{3.4}$$

Notice, however, that there is no solution if $f(y) = \cos 2\lambda y$ since then the right-hand side of (2.28) turns out to be negative. In this case the beach starts out with negative slope in contrast to the first case.

Wehausen & Laitone (1960) construct an explicit solution over a beach of a special type by integrating the differential equation arising from the bottom condition, namely $\mathrm{d}x/\mathrm{d}y = \phi_x/\phi_y$ with $x = y = 0$, using the potential

$$\phi(x,y) = \mathrm{e}^{-l_0 x}\cosh\kappa(h-y).$$

The result is

$$x = l_0\kappa^{-2}\log(\sinh\kappa h/\sinh\kappa(h-y)) \tag{3.5}$$

which describes a beach whose slope measured from the downward vertical increases monotonically from the value $\tan^{-1}(l_0/K)$ at the shoreline to $\frac{1}{2}\pi$ at large distances where the water depth is h. The beach equation (3.5) is somewhat artificial since it depends upon κ. Thus given h, and k the longshore wavenumber, any choice of κ fixes the beach slope from (3.5) since $l_0 = (k^2-\kappa^2)^{\frac{1}{2}}$, and also determines the edge wave frequency $\omega/2\pi$ from $\omega^2 = g\kappa\tanh\kappa h$. In this example the mode shape is identical to (2.26) and it remains to consider (2.28).

In terms of the notation used here we have

$$\epsilon f'(y) = \frac{l_0}{\kappa}\coth\kappa(h-y), \quad 0 \leqslant y \leqslant h \tag{3.6}$$

and it is noticeable that (3.6) satisfies (2.28) identically without approximation despite the fact that the latter has been derived under the assumption $\epsilon \ll 1$.

3.2. *Symmetric trapped waves over submerged bodies*

We seek to model trapped waves over a submerged thin body symmetric about $x = 0$. We assume that $f(y)$ vanishes except over the interval $[d-a, d+a]$ and that $f(d\pm a) = 0$.

Then from (2.29) after integrating by parts we obtain

$$l = 4K^3\epsilon \int_{d-a}^{d+a} f(y)\,\mathrm{e}^{-2Ky}\,\mathrm{d}y \tag{3.7}$$

$$= 4K^3\epsilon\, ab\,\mathrm{e}^{-2Kd} \int_{-1}^{1} F(t)\,\mathrm{e}^{-2\mu t}\,\mathrm{d}t, \quad \mu = Ka, \tag{3.8}$$

where

$$f(d+at) = bF(t) \tag{3.9}$$

so that the horizontal and vertical dimensions of the body are ϵb, and $2a$ respectively.

For example $F(t) = (1-t^2)^{\frac{1}{2}}$ corresponds to an ellipse with its centre at $(0, d)$, of minor axis $b\epsilon$, major axis a, whence (3.8) reduces to

$$l = 2\pi K^2 \epsilon b I_1(2Ka)\,\mathrm{e}^{-2Kd} \tag{3.10}$$

since

$$\int_{-1}^{1} (1-t^2)^{\frac{1}{2}}\,\mathrm{e}^{-zt}\,\mathrm{d}t = \frac{\pi}{z} I_1(z),$$

where $I_1(z)$ is the modified Bessel function.

McIver (1990) has shown, using matched asymptotic expansions that, for an ellipse which is small in both directions relative to K^{-1},

$$l = \pi K^3 \epsilon b(2a+\epsilon b)\,\mathrm{e}^{-2Kd} \tag{3.11}$$

$$\sim 2\pi K^3 a\epsilon b\,\mathrm{e}^{-2Kd}, \tag{3.12}$$

when $\epsilon b \ll a$, in agreement with (3.10) as $Ka \rightarrow 0$.

Ursell (1951) has proved the existence of trapped waves over a small submerged circular cylinder of radius a whose centre is submerged to a depth d. He showed that for sufficiently small ka, (1.3) must be satisfied for a trapped wave to exist.

To compare the result (3.10) with (1.3) we choose $a = \epsilon b$ and use $I_1(z) \sim \frac{1}{2}z$, $z \rightarrow 0$ to obtain, from (3.10),

$$l \sim 2\pi K^3 a^2\,\mathrm{e}^{-2Kd} \tag{3.13}$$

for Ka small which, since $k \sim K$, differs from (1.3) in the coefficient only. This discrepancy presumably arises from the assumption of thinness which is clearly violated by the circular cylinder. The general result for a body for which both the horizontal and vertical dimensions are small is obtained by letting $\mu = Ka \rightarrow 0$ in the integrand of (3.8) to obtain

$$l \sim 4K^3\,\mathrm{e}^{-2Kd} S, \tag{3.14}$$

where S is the area under the curve $x = \epsilon f(y)$, $y \in [d-a, d+a]$.

The corresponding result to (3.8) for finite depth is

$$l_0 = \frac{4\epsilon\kappa^3 ab}{2\kappa h + \sinh 2\kappa h} \int_{-1}^{1} F(t) \cosh 2\kappa\{h-d-at\}\,\mathrm{d}t, \tag{3.15}$$

which for both ϵb *and* a small reduces to

$$l_0 = \frac{4\kappa^3 \cosh 2\kappa(h-d)\, S}{2\kappa h + \sinh 2\kappa h}. \tag{3.16}$$

For a body on the sea bed extending upwards a distance a to a point d below the surface, so that $h-d = a$, we obtain

$$l_0 = \frac{4\epsilon\kappa^3 ab}{2\kappa h + \sinh 2\kappa h} \int_{0}^{1} F(t) \cosh 2\kappa a(1-t)\,\mathrm{d}t \tag{3.17}$$

which, if both a and ϵb are small, reduces to

$$l_0 = \frac{4\kappa^3 S}{2\kappa h + \sinh 2\kappa h}. \tag{3.18}$$

Callan (1990) has considered both the submerged circular cylinder in finite depth and the semicircular mound on the bottom using the multipole expansion method. Following Ursell's (1951) method he has shown that near the cutoff frequency, for small ka

$$l_0 = \frac{\pi\kappa^3 a^2\{3\cosh\{2\kappa(h-d)\}-1\}}{2\kappa h + \sinh 2\kappa h} \tag{3.19}$$

for the submerged circular cylinder, which agrees with Ursell's result (1.3) as $h \to \infty$, and

$$l_0 = \frac{\pi\kappa^3 a^2}{2\kappa h + \sinh 2\kappa h} \tag{3.20}$$

for the semicircular mound on the bottom.

We see that this result agrees with the approximate result (3.18) since $S = \frac{1}{4}\pi a^2$ in this case, although this can only be regarded as fortuitous.

The approximate result (3.16) when applied to the submerged circular cylinder, for which $S = \frac{1}{2}\pi a^2$, gives

$$l_0 = \frac{2\pi\kappa^3 a^2 \cosh 2\kappa(h-d)}{2\kappa h + \sinh 2\kappa h}, \tag{3.21}$$

which differs from Callan's result (3.19). Notice however that if the cylinder is resting on the bottom so that $h-d = a$, then both (3.19) and (3.21) become

$$l_0 \sim \frac{2\pi\kappa^3 a^2}{2\kappa h + \sinh 2\kappa h}.$$

3.3. *Trapped waves over a narrow shelf*

It is known that trapped waves exist over a submerged shelf, and an explicit formula exists for the longshore wavenumber k and hence l in terms of K if the shallow-water approximation is used. Thus (Evans & McIver 1984, equation (3.7)) for waves over a submerged horizontal shelf at depth d extending a horizontal distance $b\epsilon$, and of height $a = h-d$,

$$p \tan pb\epsilon = hl_0 d^{-1}, \tag{3.22}$$

where $$p = (\kappa'^2 - k^2)^{\frac{1}{2}}, \quad K = \kappa^2 h = \kappa'^2 d.$$

For ϵ small this reduces to

$$\begin{aligned} l_0 &= p^2 b\epsilon d h^{-1} \\ &= Kb\epsilon h^{-1}(1-k^2/\kappa'^2) = Kb\epsilon h^{-1}(1-k^2 d/\kappa^2 h) \end{aligned}$$

and since $k \sim \kappa$ this reduces to

$$\kappa^2 \epsilon ba/h. \tag{3.23}$$

The case of the submerged shelf can be obtained by choosing $F(t) = 1$ in (3.17) whence

$$l_0 = \frac{2b\epsilon\kappa^2 \sinh 2\kappa a}{2\kappa h + \sinh 2\kappa h} \to \frac{a\epsilon b\kappa^2}{h} \quad \text{as} \quad h \to 0, \tag{3.24}$$

in agreement with (3.23), whilst

$$l_0 \to l = 2b\epsilon K^2 \mathrm{e}^{-2Kd} \quad \text{as} \quad h \to \infty, \tag{3.25}$$

the condition for trapped waves close to the cutoff frequency over a submerged barrier of thickness $2\epsilon b$ extending from infinity to a point d beneath the surface.

The general expression (2.28) for any $f(y)$ becomes, under the shallow-water approximation,

$$l_0 = \frac{\epsilon\kappa^2}{h}\int_L (h-y) f'(y)\,\mathrm{d}y, \tag{3.26}$$

$$= KS/h^2, \tag{3.27}$$

where $S = \int_L \epsilon f(y)\,\mathrm{d}y$.

4. Trapped acoustic modes near a thin body

The arguments leading to the result (2.25) can also be used to derive corresponding results for the following problem. The thin body $x = \epsilon f(y)$, symmetric about $x = 0$ is contained between boundaries $y = 0$ and $y = h$ enclosing a medium in which the potential satisfies

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0. \tag{4.1}$$

This could be an acoustic medium, when $k = \omega/c$ and c is the velocity of sound, or (4.1) could describe a water-wave problem in a channel of depth H with a vertical cylinder $x = \epsilon f(y)$, $0 \leqslant z \leqslant H$ extending throughout the depth. In this case (1.5) is satisfied and $k = \kappa$.

In order to obtain trapped modes we impose the conditions

$$\phi_y = 0, \quad y = 0, \tag{4.2}$$

$$\phi = 0, \quad y = h, \tag{4.3}$$

and consider possible solutions in $x > 0$, $0 \leqslant y \leqslant h$.

These imply motion in a channel of width $2h$ which is antisymmetric about the centreplane, but symmetric about $x = 0$, satisfying a no-flow condition on the channel walls, in the presence of the cylinder plus its image in $y = h$. Alternatively, by reflection in $y = 0$, they describe motion in a channel of width $2h$ with the soft condition $\phi = 0$ on the walls.

The result (2.25) now follows as before except that in this case

$$\psi_n(y) = (2/h)^{\frac{1}{2}} \cos k_n y, \quad k_n = (n+\tfrac{1}{2})\pi/h \tag{4.4}$$

and $l_n = (k_n^2 - k^2)^{\frac{1}{2}}$, $n = 0, 1, 2, \ldots$ with $k < k_0$. Then it follows from (4.4) that (2.25) can be written

$$l_0 = \frac{\epsilon k_0}{h}\int_L f'(y)\sin 2k_0 y\,\mathrm{d}y \tag{4.5}$$

$$= \frac{2\epsilon k_0^2}{h}\int_L f(y)\cos 2k_0 y\,\mathrm{d}y. \tag{4.6}$$

As a simple example the plane $x = \epsilon y$, $0 \leqslant y \leqslant h$ gives rise to the simple relation

$$l_0 h = \epsilon. \tag{4.7}$$

The result equivalent to (3.15) for a cylinder on $y = h$ together with its image in $y = h$ is

$$l_0 = \frac{2\epsilon k_0^2 ab}{h}\int_0^1 F(y)\cos 2k_0 a(1-t)\,\mathrm{d}t \tag{4.8}$$

and the same notation as (3.9) *et seq.* is used.

As a check on this result we consider a thin rectangular block, so that $F(t) = 1$ whence (4.8) reduces to

$$l_0 h = k_0 \epsilon b \sin 2k_0 a, \quad k_0 = \pi/2h, \tag{4.9}$$

in agreement with recent results of Evans & Linton (1990) who obtain (4.9) as a limiting case of a general formulation for arbitrarily sized rectangular blocks in channels.

Another example is the thin elliptical cylinder in the channel. By putting $F(t) = (1-t^2)^{\frac{1}{2}}$ in (4.8) we obtain

$$l_0 = \frac{2\epsilon k_0^2 ab}{h} I, \tag{4.10}$$

where
$$I = \frac{\pi}{4k_0 a}(\cos(2k_0 a) J_1(2k_0 a) + \sin(2k_0 a) H_1(2k_0 a)) \tag{4.11}$$

and J_1 is a Bessel function and H_1 a Struve function.

McIver (1990) obtains for a small ellipse in a channel,

$$l_0 = \epsilon b(a+\epsilon b) k_0^3 \tag{4.12}$$

$$\sim \epsilon b a k_0^3, \tag{4.13}$$

in agreement with (4.11) as $k_0 a \to 0$.

5. Conclusion

In this paper explicit forms have been derived for the relation between the longshore wavenumber k and the wave frequency ω, for waves travelling near the cutoff frequency over submerged thin bodies, near-vertical beaches or thin vertical cylinders in a channel, each characterized by the thinness parameter ϵ. The results agree with known results for edge waves over a nearly vertical cliff and shallow-water trapped waves over a narrow rectangular shelf, and for antisymmetric modes in a channel containing a thin vertical rectangular or elliptical cylinder.

It is possible to extend the idea to consider trapped waves near the lowest cutoff frequency in a narrow long wave tank, containing a thin three-dimensional submerged body which for $\epsilon = 0$ reduces to a vertical lamina perpendicular to the tank walls. Thus Callan (1990) has extended equations (2.28) and (2.29) to cover this case, and has found exact agreement between this approximate theory and a theory based on multipole expansions for the connection between longshore wavenumber and wavelength in the cases of a submerged sphere in both finite or infinitely deep water, when simultaneously the wave frequency approaches the first cutoff frequency and the size of the sphere shrinks to zero. An alternative approach by McIver (1990) using matched asymptotic expansions, has produced equivalent expressions for bodies which are small in all directions in the submerged horizontal cylinder case, and both the vertical cylinder and the three-dimensional body in a channel.

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